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ANALYSIS OF FREE VIBRATIONS FOR RAYLEIGH — LAMB WAVES IN A MICROSTRETCH THERMOELASTIC PLATE WITH TWO RELAXATION TIMES

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The propagation of free vibrations in a microstretch thermoelastic homogeneous isotropic plate subjected to stress-free thermally insulated and isothermal conditions is investigated in the context of conventional coupled thermoelasticity (CT) and Green and Lindsay (G—L) theories of thermoelasticity. The secular equations for the microstretch thermoelastic plate in closed form for symmetric and skew-symmetric wave mode propagation in completely separate terms are derived. At short wavelength limits, the secular equations for both modes in a stress-free thermally insulated and isothermal homogeneous isotropic microstretch thermoelastic plate reduce to the Rayleigh surface wave frequency equation. The results for symmetric and skew-symmetric wave modes are computed numerically and presented graphically. The theory and numerical computations are found to be in close agreement.

Keywords: *Microstretch, secular equations, phase velocity, attenuation.*

Introduction. The mechanical behavior of solids with voids (e.g., porous solids) and solids containing microscopic components (e.g., nanocomposites) cannot be described by means of the classical theory of elasticity. Important among such materials are animal bones, solids with microcracks, foams, and other synthetic materials with pores and/or microreinforcements.

A microcontinuum is considered as the collection of material particles which can deform independently in the microscale in addition to the classical bulk deformation of the material. Eringen and Suhubi [1, 2] introduced and developed a general theory for this phenomenon, which is called micromorphic continua. As is known, the general micromorphic theory is very complicated even for linear case. To overcome the difficulties, Eringen was the first to introduce the micropolar elasticity [3] and then the microstretch elasticity [4]. A microstretch elastic solid possesses seven degrees of freedom: three for translation, three for rotation (as in micropolar elasticity), and a stretch (breathing type motion) required for the substructures.

Generalized thermoelasticity theories have been developed with the objective of removing the paradox of infinite speed of heat propagation inherent in the conventional coupled dynamical theory of thermoelasticity, where the parabolic type heat conduction equation is based on Fourier's law of heat conduction. This newly emerged theory, which admits finite speed of heat propagation, is now referred to as the hyperbolic thermoelasticity theory [5], since the heat equation for a rigid conductor is a hyperbolic differential equation.

There are two important generalized theories of thermoelasticity. The first is due to Lord and Shulman [6]. The second generalization to the coupled theory of elasticity is known as the theory of thermoelasticity with two relaxation times or the theory of temperature — rate-dependent thermoelasticity. Muller [7] in a review of the thermodynamics of thermoelastic solids proposed an entropy production inequality, with the help of which he considered restrictions on a class of constitutive equations. A generalization of this inequality was proposed by Green and Laws [8]. Green and Lindsay obtained another version of the constitutive equations [9]. These equations were also obtained independently and more explicitly by Suhubi [10]. This theory contains two constants that act as relaxation times and modify all the equations of the coupled theory, not only the heat equations. The classical Fourier law of heat conduction is not violated if the medium under consideration has a center of symmetry.

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Kumar and Singh [11] studied wave propagation in a generalized thermo-microstretch elastic solid. Kumar and Deswal [12] investigated wave propagation through a cylindrical bore contained in a microstretch elastic medium. De Cicco [13] considered the stress concentration effects in microstretch elastic bodies. Liu and Hu [14] investigated the problem of microstretch inclusion. Svanadze [15] constructed fundamental solution of the system of equations of steady oscillations in the theory of microstretch elastic solids. Kumar and Partap [16] investigated reflection of plane waves in a heat flux-dependent microstretch thermoelastic solid half-space.

The present investigation is concerned with the propagation of waves in an infinite homogeneous isotropic microstretch generalized thermoelastic plate of finite thickness. More general dispersion equations of microstretch thermoelastic Lamb type waves are derived and discussed. The secular equations for different conditions of solutions are deduced. Numerical solutions of the dispersion equations and attenuation coefficients for the symmetric and skew-symmetric modes are presented graphically.

Basic Equations. The equations of motion for the linear theory of microstretch thermoelasticity that are given by Eringen [17] are

$$t_{kl,k} + \rho\,(f_l - \ddot{u}_l) = 0\,, \tag{1}$$

$$m_{kl,k} + \varepsilon_{lmn} t_{mn} = \rho j_{lm} \ddot{\phi}_m\,, \tag{2}$$

$$m_{k,k} + g - s = \frac{1}{2}\rho j_0 \ddot{\phi}^*\,. \tag{3}$$

The constitutive relations in a homogeneous isotropic microstretch thermoelastic solid in the absence of body forces, body couples, stretch force, and heat sources are

$$t_{kl} = \lambda u_{r,r}\delta_{kl} + \mu\,(u_{k,l} + u_{l,k}) + K\,(u_{l,k} - \varepsilon_{klr}\phi_r) - \nu\left(T + \tau_1 \frac{\partial T}{\partial t}\right)\delta_{kl} + \lambda_0 \delta_{kl}\phi^*\,, \tag{4}$$

$$m_{kl} = a\phi_{r,r}\delta_{kl} + \beta\phi_{k,l} + \gamma\phi_{l,k} + b_0\varepsilon_{mlk}\phi^*_{,m}\,, \tag{5}$$

$$s - g = -\nu_1\left(T + \tau_1 \frac{\partial T}{\partial t}\right) + \lambda_1\phi^* + \lambda_0 \nabla\cdot\mathbf{u}\,, \tag{6}$$

$$m_k = \alpha_0 \phi^*_{,k} + b_0 \varepsilon_{klm}\phi_{l,m}\,, \tag{7}$$

where $\nu = (3\lambda + 2\mu + K)\alpha_{t1}$, $\nu_1 = (3\lambda + 2\mu + K)\alpha_{t2}$ and the comma in the subscripts denotes spatial derivatives.

Using the constitutive relations (4)—(7) in Eqs. (1)—(3), we get the following equations in vectorial form:

$$(\lambda + 2\mu + K)\,\nabla\,(\nabla\cdot\mathbf{u}) - (\mu + K)\,\nabla\times\nabla\times\mathbf{u} + K\nabla\times\phi - \nu\left(1 + \tau_1 \frac{\partial}{\partial t}\right)\nabla T + \lambda_0\nabla\phi^* = \rho\,\frac{\partial^2\mathbf{u}}{\partial t^2}\,, \tag{8}$$

$$(\alpha + \beta + \gamma)\,\nabla\,(\nabla\cdot\boldsymbol{\phi}) - \gamma\nabla\times(\nabla\times\boldsymbol{\phi}) + K\nabla\times\mathbf{u} - 2K\phi = \rho j\,\frac{\partial^2\boldsymbol{\phi}}{\partial t^2}\,, \tag{9}$$

$$\alpha_0\nabla^2\phi^* + \nu_1\left(T + \tau_1 \frac{\partial T}{\partial t}\right) - \lambda_1\phi^* - \lambda_0\nabla\cdot\mathbf{u} = \frac{\rho j_0}{2}\,\frac{\partial^2\phi^*}{\partial t^2}\,. \tag{10}$$

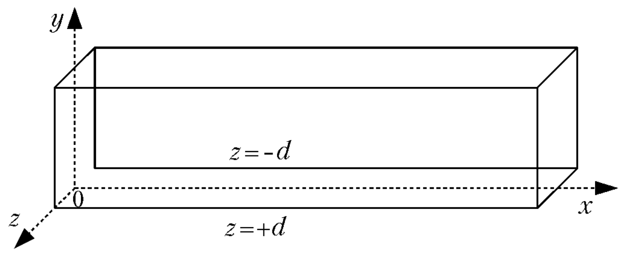

Fig. 1. Geometry of the problem.

According to Eringen [17] and Green and Lindsay [9], the heat equation is given as

$$K^*\nabla^2 T = \rho C^* \left(\frac{\partial T}{\partial t} + \tau_0 \frac{\partial^2 T}{\partial t^2}\right) + \nu T_0 \frac{\partial}{\partial t}(\nabla\cdot\mathbf{u}) + \nu_1 T_0 \frac{\partial \phi^*}{\partial t} . \tag{11}$$

Formulation and Solution of the Problem. We consider an infinite homogeneous isotropic thermally conducting microstretch elastic plate of thickness $2d$ initially undisturbed and found at uniform temperature T_0. The plate is axisymmetric with the z axis as the axis of symmetry (see Fig. 1). The origin of the coordinate system (x, y, z) is taken as the middle surface of the plate. The surfaces $z = \pm d$ are subjected to different boundary conditions. The x–y plane is chosen to coincide with the middle surface, and the z axis is normal to it.

For a two-dimensional problem, we take

$$\mathbf{u} = (u_1, 0, u_3) , \quad \boldsymbol{\phi} = (0, \phi_2, 0) \tag{12}$$

and define the nondimensional quantities

$$x' = \frac{\omega^* x}{c_1}, \quad z' = \frac{\omega^* z}{c_1}, \quad u_1' = \frac{\rho\omega^* c_1}{\nu T_0} u_1 , \quad u_3' = \frac{\rho\omega^* c_1}{\nu T_0} u_3 , \quad t' = \omega^* t , \quad \phi_2' = \frac{\rho c_1^2}{\nu T_0}\phi_2 , \quad \phi'^* = \frac{\rho c_1^2}{\nu T_0}\phi^* , \quad T' = \frac{T}{T_0} ,$$

$$\tau_1' = \omega^*\tau_1 , \quad \tau_0' = \omega^*\tau_0 , \quad t_{ij}' = \frac{1}{\nu T_0} t_{ij} , \quad m_{ij}' = \frac{\omega^* m_{ij}}{c_1 \nu T_0} , \quad h' = \frac{c_1 h}{\omega^*} , \quad c_1 = \sqrt{\frac{\lambda + 2\mu + K}{\rho}} ,$$

$$c_2 = \sqrt{\frac{\mu + K}{\rho}} , \quad c_3^2 = \frac{\gamma}{\rho j} , \quad c_4^2 = \frac{2\alpha_0}{\rho j_0} , \quad p = \frac{K}{\rho c_1^2} , \quad p_1 = \frac{\lambda_1}{\rho c_1^2} , \quad p_0 = \frac{\lambda_0}{\rho c_1^2} , \quad \delta^2 = \frac{c_2^2}{c_1^2} , \quad \delta_1^2 = \frac{c_3^2}{c_1^2} , \quad \delta_2^2 = \frac{c_4^2}{c_1^2} ,$$

$$\delta^* = \frac{K c_1^2}{\gamma \omega^{*2}} , \quad \delta_1^* = \frac{\rho c_1^4}{\alpha_0 \omega^{*2}} , \quad G^* = \frac{\nu^2 T_0}{\rho C^* (\lambda + 2\mu + K)} , \quad \omega^* = \frac{\rho C^* c_1^2}{K^*} , \quad \bar{\nu} = \frac{\nu_1}{\nu} , \quad m_k' = \frac{\omega^* m_k}{c_1 \nu T_0} . \tag{13}$$

With the help of these quantities, Eqs. (8)–(11) can be presented in the following nondimensional form (after omitting the primes for convenience):

$$(1 - \delta^2)\left(\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_3}{\partial x \partial z}\right) + \delta^2\nabla^2 u_1 + p_0 \frac{\partial \phi^*}{\partial x} - p\frac{\partial \phi_2}{\partial z} - \left(1 + \tau_1 \frac{\partial}{\partial t}\right)\frac{\partial T}{\partial x} = \frac{\partial^2 u_1}{\partial t^2} , \tag{14}$$

$$(1 - \delta^2)\left(\frac{\partial^2 u_1}{\partial x \partial z} + \frac{\partial^2 u_3}{\partial z^2}\right) + \delta^2\nabla^2 u_3 + p_0 \frac{\partial \phi^*}{\partial z} + p\frac{\partial \phi_2}{\partial x} - \left(1 + \tau_1 \frac{\partial}{\partial t}\right)\frac{\partial T}{\partial z} = \frac{\partial^2 u_3}{\partial t^2} , \tag{15}$$

$$\nabla^2\phi_2 + \delta^{*^2}\left(\frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x}\right) - 2\delta^{*^2}\phi_2 = \frac{1}{\delta_1^2}\frac{\partial^2\phi_2}{\partial t^2}\,, \tag{16}$$

$$\nabla^2\phi^* + \bar{\nu}\delta_1^*\left(T + \tau_1\frac{\partial T}{\partial t}\right) - p_1\delta_1^*\phi^* - p_0\delta_1^*\left(\frac{\partial u_1}{\partial x} + \frac{\partial u_3}{\partial z}\right) = \frac{1}{\delta_2^2}\frac{\partial^2\phi^*}{\partial t^2}\,, \tag{17}$$

$$\nabla^2 T - \left(\frac{\partial T}{\partial t} + \tau_0\frac{\partial^2 T}{\partial t^2}\right) = \bar{\nu}G^*\frac{\partial\phi^*}{\partial t} + G^*\frac{\partial}{\partial t}\left(\frac{\partial u_1}{\partial x} + \frac{\partial u_3}{\partial z}\right). \tag{18}$$

Introducing the potential functions ϕ and ψ into Eqs. (14)–(18) through the relations

$$u_1 = \frac{\partial\phi}{\partial x} + \frac{\partial\psi}{\partial z}\,,\quad u_3 = \frac{\partial\phi}{\partial z} + \frac{\partial\psi}{\partial x}\,, \tag{19}$$

we obtain

$$\left(\nabla^2 - \frac{\partial^2}{\partial t^2}\right)\phi + p_0\phi^* - \left(T + \tau_1\frac{\partial T}{\partial t}\right) = 0\,, \tag{20}$$

$$\nabla^2\psi - \frac{p\phi_2}{\delta^2} - \frac{1}{\delta^2}\frac{\partial^2\psi}{\partial t^2} = 0\,, \tag{21}$$

$$\nabla^2\phi_2 + \delta^{*^2}\nabla^2\psi - 2\delta^{*^2}\phi_2 - \frac{1}{\delta_1^2}\frac{\partial^2\phi_2}{\partial t^2} = 0\,, \tag{22}$$

$$\nabla^2\phi^* - p_1\delta_1^*\phi^* - p_0\delta_1^*\nabla^2\phi + \bar{\nu}\delta_1^*\left(T + \tau_1\frac{\partial T}{\partial t}\right) - \frac{1}{\delta_2^2}\frac{\partial^2\phi^*}{\partial t^2} = 0\,, \tag{23}$$

$$\nabla^2 T - (\dot{T} + \tau_0\ddot{T}) = G^*\nabla^2\dot{\phi} + \bar{\nu}G^*\frac{\partial\phi^*}{\partial t}\,. \tag{24}$$

Equations (20), (23), and (24) are coupled equations relative to ϕ, T, ϕ^*, whereas Eqs. (21) and (22) are coupled ones with respect to ψ and ϕ_2.

We consider the propagation of plane waves in the *x*—*z* plane with the wavefront parallel to the *y* axis; therefore, ϕ, T, ϕ^*, ψ, and ϕ_2 are independent of *y*. We assume the solutions of Eqs. (20)–(24) in the form

$$(\phi, \psi, \phi_2, T, \phi^*) = [f(z), g(z), w(z), h(z), \eta(z)]\exp\{i\xi(x - ct)\}\,, \tag{25}$$

where $c = \omega/\xi$. Using Eq. (25) in Eqs. (20)—(24), we obtain after a number of transformations

$$\left[\nabla_1^6 + A\nabla_1^4 + B\nabla_1^2 + C\right]f(z) = 0\,, \tag{26}$$

$$\left[\nabla_1^4 + E\nabla_1^2 + F\right] g(z) = 0\,, \tag{27}$$

where

$$\nabla_1^2 = \frac{d^2}{dz^2} - \xi^2\,,$$

$$A = 1 + k_0 + \frac{1}{\delta_2^2} - \frac{p_1\delta_1^*}{\omega^2} + \frac{p_0^2\delta_1^*}{\omega^2} - i\omega G^* k_1 k_0'\,,$$

$$B = k_0 + \frac{1 + k_0}{\delta_2^2} - \frac{\delta_1^*}{\omega^2}\left[(1 + k_0)\, p_1 - p_0^2 k_0 + i\omega G^* k_0'\, k_1\, (\bar{\nu} p_0 - p_1) - i\omega G^* k_0'\, k_1 \bar{\nu}\, (\bar{\nu} - p_0)\right] - \frac{i\omega G^* k_0'\, k_1}{\delta_2^2}\,,$$

$$C = k_0\left(\frac{1}{\delta_2^2} - \frac{p_1\delta_1^*}{\omega^2}\right) - \frac{ik_0'\, k_1 G^* \delta_1^* \bar{\nu}^2}{\omega}\,,$$

$$E = \frac{1}{\delta^2} + \frac{1}{\delta_1^2} + \frac{\delta^*\,(p - 2\delta^2)}{\omega^2\delta^2}\,,$$

$$F = \frac{1}{\delta^2}\left(\frac{1}{\delta_1^2} - \frac{2\delta^*}{\omega^2}\right),$$

$$k_0 = \tau_0 + i\omega^{-1}\,, \quad k_0' = i\omega^{-1}\,, \quad k_1 = \tau_1 + i\omega^{-1}\,.$$

Equation (26) can be rewritten as

$$\left[\nabla^{*3} + A\nabla^{*2} + B\nabla^* + C\right] f(z) = 0\,, \tag{28}$$

where $\nabla^* = \nabla_1^2$.

Using Cardon's method to solve Eq. (28), we obtain

$$s^3 + 3Hs + G = 0\,, \tag{29}$$

where $s = \nabla^* + \frac{A}{3}, H = \frac{B}{3} - \frac{A^2}{9}, G = \frac{2A^3}{27} - \frac{AB}{3} + C.$

In order that all the three roots of Eq. (29) be real, $\Delta = G^2 + 4H^3$ should be negative. Assuming that Δ is negative*, we obtain the three roots of Eq. (29) as

$$s_n = 2\,(-H)^{1/2} \cos\frac{\phi + 2\pi\,(n-1)}{3}\,, \quad n = 1, 2, 3\,, \tag{30}$$

where $\phi = \arctan(|\Delta|^{1/2} - G)$. Hence,

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*) From the Editors

The referee considers that the calculation of the discriminant can be carried out only for concrete numerical data that cha- racterize the micropolar medium with microstretch, so that the assumption as to negative discriminant is not justi-

$$\nabla_n^* = s_n - \frac{A}{3} . \tag{31}$$

Equation (31) can be written as

$$\left(\frac{d^2}{dz^2} - m_n^2 \right) f(z) = 0 , \tag{32}$$

where $m_n^2 = \xi^2 + s_n - \frac{A}{3}$ for n = 1, 2, 3; m_1, m_2, and m_3 correspond to longitudinal displacement wave, thermal wave, and longitudinal microstretch wave, respectively. For n = 4, 5 we have

$$m_{4,5}^2 = \frac{-E \pm \sqrt{E^2 - 4F}}{2} .$$

The appropriate potentials will be obtained as

$$\phi = (A_1 \cos m_1 z + A_2 \cos m_2 z + A_3 \cos m_3 z + B_1 \sin m_1 z + B_2 \sin m_2 z + B_3 \sin m_3 z) \exp \{ i\xi (x - ct) \} ,$$

$$\psi = (A_4 \cos m_4 z + B_4 \sin m_4 z + A_5 \cos m_5 z + B_5 \sin m_5 z) \exp \{ i\xi (x - ct) \} ,$$

$$\phi_2 = \frac{\delta^2}{p} \left[(b^2 - m_4^2)(A_4 \cos m_4 z + B_4 \sin m_4 z) + (b^2 - m_5^2)(A_5 \cos m_5 z + B_5 \sin m_5 z) \right] \exp \{ i\xi (x - ct) \} ,$$

$$T = \left[S_1 (A_1 \cos m_1 z + B_1 \sin m_1 z) + S_2 (A_2 \cos m_2 z + B_2 \sin m_2 z) + S_3 (A_3 \cos m_3 z + B_3 \sin m_3 z) \right] \times \exp \{ i\xi (x - ct) \} ,$$

$$\phi^* = \left[V_1 (A_1 \cos m_1 z + B_1 \sin m_1 z) + V_2 (A_2 \cos m_2 z + B_2 \sin m_2 z) + V_3 (A_3 \cos m_3 z + B_3 \sin m_3 z) \right] \times \exp \{ i\xi (x - ct) \} , \tag{33}$$

where

$$b^2 = \xi^2 \left(\frac{c^2}{\delta^2} - 1 \right), \quad V_i = \frac{-\delta_1^* \left[(\bar{\nu} - p_0) m_i^2 - \bar{\nu}\xi^2 \left(c^2 + \frac{p_0}{\bar{\nu}} - 1 \right) \right]}{m_i^2 - \xi^2 \left[c^2 \left(\frac{1}{\delta_2^2} - \frac{\delta_1^*}{\omega^2} (\bar{\nu} p_0 - p_1) \right) - 1 \right]} ,$$

$$S_i = \frac{\left[m_i^2 - \xi^2 (c^2 - 1) \right] \left[m_i^2 - \xi^2 \left(c^2 \left(\frac{1}{\delta_2^2} - \frac{\delta_1^*}{\omega^2} (\bar{\nu} p_0 - p_1) \right) - 1 \right) \right] + \delta_1^* p_0 \left[(\bar{\nu} - p_0) m_i^2 - \bar{\nu}\xi^2 \left(c^2 + \frac{p_0}{\bar{\nu}} - 1 \right) \right]}{i\omega k_1 \left[m_i^2 - \xi^2 \left(c^2 \left(\frac{1}{\delta_2^2} - \frac{\delta_1^*}{\omega^2} (\bar{\nu} p_0 - p_1) \right) - 1 \right) \right]} ,$$

$$i = 1, 2, 3 .$$

Boundary Conditions. We consider the following nondimensional mechanical and thermal boundary conditions at the plate surfaces $z = \pm d$.

Mechanical boundary conditions are given as follows:

$$t_{33}=0\,,\quad t_{31}=0\,,\quad m_{32}=0\,,\quad m_{,3}=0\,. \tag{34}$$

where

$$t_{33}=(\lambda+2\mu+K)\frac{\partial u_3}{\partial z}+\lambda\frac{\partial u_1}{\partial x}-\nu\left(T+\tau_1\frac{\partial T}{\partial t}\right)+\lambda_0\phi^*\,,\quad t_{31}=(\mu+K)\frac{\partial u_1}{\partial z}+\mu\frac{\partial u_3}{\partial x}-K\phi_2\,,$$

$$m_{32}=\gamma\frac{\partial \phi_2}{\partial z}+b_0\frac{\partial \phi^*}{\partial x}\,,\quad m_{,3}=\alpha_0\frac{\partial \phi^*}{\partial z}-b_0\frac{\partial \phi_2}{\partial x}\,.$$

The thermal boundary condition takes the form

$$T_{,z}+hT=0\,, \tag{35}$$

where $h\to 0$ corresponds to thermal insulated boundaries and $h\to\infty$ refers to isothermal one.

Derivation of the Secular Equations. Using the boundary conditions (34) and (35), with the help of Eqs. (33) we obtain a system of ten equations. This system has a nontrivial solution if the determinant of the coefficients at amplitudes vanishes. After lengthy algebraic reductions and manipulations we obtain cumbersome secular equations for both limiting cases: insulated boundaries ($h\to 0$) and isothermal ones ($h\to\infty$). For brevity, we present only the first of these equations:

$$\left[\frac{T_1}{T_4}\right]^{\pm 1}-\frac{m_1\,(V_1S_3-V_3S_1)}{m_2\,(V_2S_3-V_3S_2)}\left[\frac{T_2}{T_4}\right]^{\pm 1}+\frac{m_1\,(V_1S_2-V_2S_1)}{m_3\,(V_2S_3-V_3S_2)}\left[\frac{T_3}{T_4}\right]^{\pm 1}$$

$$+\frac{RV}{SU}\,\frac{f_5-f_4}{m_5f_5T_4-m_4f_4T_5}\,\frac{m_1S_1\,(V_3-V_2)}{m_2m_3\,(V_2S_3-V_3S_2)}$$

$$\times\left\{\left[\frac{T_2T_3}{T_4^2T_5}\right]^{\pm 1}-\frac{m_2S_2\,(V_3-V_1)}{m_1S_1\,(V_3-V_2)}\left[\frac{T_1T_3}{T_4^2T_5}\right]^{\pm 1}+\frac{m_3S_3\,(V_2-V_1)}{m_1S_1\,(V_3-V_2)}\left[\frac{T_1T_2}{T_4^2T_5}\right]^{\pm 1}\right\}$$

$$+\frac{Q}{P}\,\frac{m_4-m_5\left(\frac{T_5}{T_4}\right)^{\pm 1}}{m_5f_5-m_4f_4\left(\frac{T_5}{T_4}\right)^{\pm 1}}\left\{\frac{V}{U}\,\frac{f_5f_4\,(S_3-S_2)}{(V_2S_3-V_3S_2)}\left[\left[\frac{T_1}{T_4}\right]^{\pm 1}-\frac{m_1\,(S_3-S_1)}{m_2\,(S_3-S_2)}\left[\frac{T_2}{T_4}\right]^{\pm 1}+\frac{m_1\,(S_2-S_1)}{m_3\,(S_3-S_2)}\left[\frac{T_3}{T_4}\right]^{\pm 1}\right]\right.$$

$$\left.+\frac{R}{S}\left[\left[\frac{T_1}{T_4}\right]^{\pm 1}-\frac{m_1V_2\,(V_1S_3-V_3S_1)}{m_2V_1\,(V_2S_3-V_3S_2)}\left[\frac{T_2}{T_4}\right]^{\pm 1}+\frac{m_1V_3\,(V_1S_2-V_2S_1)}{m_3V_1\,(V_2S_3-V_3S_2)}\left[\frac{T_3}{T_4}\right]^{\pm 1}\right]\right\}$$

$$=\frac{-Q^2m_1m_4m_5\,[(V_1-V_2)\,(S_2-S_3)-(V_2-V_3)\,(S_1-S_2)]\,(f_5-f_4)}{P^2\,(V_2S_3-V_3S_2)\left(m_5f_5-m_4f_4\left(\frac{T_5}{T_4}\right)^{\pm 1}\right)}-$$

$$-\frac{Q^2}{P^2}\frac{RV}{SU}\frac{m_5 f_4 T_4 - m_4 f_5 T_5}{m_5 f_5 T_4 - m_4 f_4 T_5}\frac{V_1 (S_3 - S_2)}{V_2 S_3 - V_3 S_2}$$

$$\times\left\{\left[\frac{T_1}{T_4}\right]^{\pm 1} - \frac{m_1 V_2 (S_3 - S_1)}{m_2 V_1 (S_3 - S_2)}\left[\frac{T_2}{T_4}\right]^{\pm 1} + \frac{m_1 V_3 (S_2 - S_1)}{m_3 V_1 (S_3 - S_2)}\left[\frac{T_3}{T_4}\right]^{\pm 1}\right\}, \tag{36}$$

where

$$P = b^2 - \xi^2 + \frac{p\xi^2}{\delta^2}, \quad Q = -2i\xi\left(1 - \frac{p}{2\delta^2}\right), \quad f_i = b^2 - m_i^2, \quad i = 4, 5; \quad R = i\xi b_0, \quad S = \frac{\gamma\delta^2}{p},$$

$$U = \alpha_0, \quad V = \frac{b_0 i\xi\delta^2}{p}; \quad T_i = \tan m_i d, \quad i = 1, 2, 3, 4, 5.$$

Here, the superscripts +1 and –1 refer to the skew-symmetric and symmetric modes, respectively. Equation (36) and an omitted equation for isothermal boundaries are the most general dispersion relations involving the wave number and phase velocity of various modes of propagation in a microstretch generalized thermoelastic plates under different situations. They can be recognized as the modified Rayleigh–Lamb equations which respectively describe the symmetric and antisymmetric modes of wave propagation for a force-stress and couple-stress free, thermally insulated, and isothermal microstretch thermoelastic plate.

Particular cases. In the case of *a microstretch coupled thermoelastic plate* (with coupled theory of thermoelasticity (CT)), the thermal relaxation times vanish, i.e., $\tau_0 = \tau_1 = 0$, so that $k_0 = k_0' = k_1 = i\omega^{-1}$.
In the *absence of the microstretch effect*, we have for *a micropolar thermoelastic plate*

$$R = V = U = V_1 = V_3 = 0, \; V_2 = 1, \; S_2 = 0, \; S_i = i\omega^{-1}k_1^{-1}(m_i^2 - a^2), \; i = 1, 3, \text{ where } a^2 = \xi^2(c^2 - 1).$$

In the *absence of a micropolarity effect* ($K = p = 0$) we obtain for *a thermoelastic plate* $m_4^2 = b_2, m_5^2 = \xi^2(\frac{c^2}{\delta_1^2} - 1)$.

In the *absence of a thermal effect*, the mentioned secular equations for *the elastic plate* reduce to equations that agree with the ones presented by Graff [19].

Different regions of the secular equation are considered. Depending upon whether m_1, m_2, m_3, m_4, m_5, b being real, purely imaginary, or complex, Eq. (36) is correspondingly altered. For example, when the characteristic roots are of the type $a^2 = -a'^2$, $b^2 = -b'^2$, $m_k^2 = -a_k^2$ (k = 1, 2, 3, 4, 5), we have $a = ia', b = ib', m_k = i\, a_k$ which are purely imaginary or complex numbers. This ensures that the superposition of partial waves has the property of exponential decay. In this case, the secular equations are obtained by replacing the circular tangent functions of m_k with hyperbolic tangent functions of a_k.

We also consider the case of *a thin plate*, when the transverse wavelength with respect to the thickness of the plate is quite large, and the case of short wavelength, when this relation is quite small. Moreover, a special class of exact solutions, called the Lamé modes, but evidently first identified by Lamb [18], can be obtained by considering the special case $b^2 = \xi^2(1 - \frac{p}{\delta^2})$. Here, the frequency is given by

$$\omega = \frac{\sqrt{4b^2d^2 + n^2}\,\pi^2\left(1 - \frac{p}{\delta^2}\right)^2}{2da_4\sqrt{1 - \frac{p}{\delta^2}}}. \tag{37}$$

It is obvious that these modes depend upon the micropolar parameter (K or p) and the plate thickness. However, in the absence of stretch and micropolarity effect the frequency is given as $\omega = \frac{n\pi\delta}{\sqrt{2}d}$, which agrees with the result of Graff [19].

Numerical Results and Discussion. With the view of illustrating the theoretical results obtained in the preceding sections and comparing these for various theories of thermoelasticity, we present some numerical results. The material chosen for this purpose is magnesium crystal (a microstretch thermoelastic solid). The micropolar parameters are the following [20]:

$$\rho = 1.74 \cdot 10^3 \text{ kg/m}^3, \quad \lambda = 9.4 \cdot 10^{10} \text{ N/m}^2, \quad \mu = 4.0 \cdot 10^{10} \text{ N/m}^2,$$

$$K = 10^{10} \text{ N/m}^2, \quad \gamma = 0.779 \cdot 10^{-9} \text{ N}, \quad j = 0.2 \cdot 10^{-19} \text{ m}^2, \quad j_0 = 0.185 \cdot 10^{-19} \text{ m}^2.$$

The thermal characteristics were taken from [21]:

$$\tau_0 = 6.131 \cdot 10^{-13} \text{ sec}, \quad \tau_1 = 8.765 \cdot 10^{-13} \text{ sec}, \quad G^* = 0.028, \quad T_0 = 23^\circ\text{C}, \quad C^* = 1.04 \cdot 10^3 \text{ J/(kg·deg)},$$

$$K^* = 1.7 \cdot 10^6 \text{ J/(m·sec·deg)}, \quad \nu = 2.68 \cdot 10^6 \text{ N/(m}^2\text{·deg)}, \quad \nu_1 = 2.0 \cdot 10^6 \text{ N/(m}^2\text{·deg)}$$

and the stretch parameters from [22]:

$$\lambda_0 = 0.5 \cdot 10^{10} \text{ N/m}^2, \quad \lambda_1 = 0.5 \cdot 10^{10} \text{ N/m}^2, \quad \alpha_0 = 0.779 \cdot 10^{-9} \text{ N}, \quad b_0 = 0.5 \cdot 10^{-9} \text{ N}, \quad d = 0.01 \text{ m}.$$

In general, the wave number and phase velocity of the waves are complex quantities; therefore, the waves are attenuated in space. We can write

$$c^{-1} = v^{-1} + i\omega^{-1} q; \tag{38}$$

then $\xi = K_1 + iq$, where $K_1 = \omega/v$ and q are real numbers. This shows that v is the propagation speed and q is the attenuation coefficient of waves. Based on Eq. (38), a FORTRAN program has been developed for the solution of Eq. (36), so that the phase velocity c and the attenuation coefficient q for different modes of wave propagation can be obtained. Here the following relations are used: $\tan\theta = \tan(n\pi + \theta)$, $m_n^2 = \xi^2 + s_n - \frac{A}{3}$.

The nondimensional phase velocity and attenuation coefficient of the symmetric and skew-symmetric modes of wave propagation in the context of the Green and Lindsay (G – L) and CT theories of thermoelasticity have been computed for various values of the nondimensional wave number from dispersion equation (36) for stress-free thermally insulated boundaries. The results are presented graphically for different modes (n = 0, 1, 2) as functions of the wave number in Fig. 2.

The phase velocities of the lowest symmetric mode of propagation become dispersionless, i.e., they remain constant with variation in the wave number (Fig. 2a). The phase velocities of higher modes of propagation in the symmetric and skew-symmetric cases (Fig. 2a and b) attain quite large values at low wave numbers, then sharply decrease, become steady, and asymptotically tend to the reduced Rayleigh wave velocity with increasing wave number. The reason for this asymptotic approach is that for short wavelengths (or high frequencies) the material plate behaves increasingly like a thick slab, hence the coupling between the upper and lower boundary surfaces is reduced and as a result the properties of the symmetric and skew-symmetric waves become more and more similar. It is observed that for various theories of thermoelasticity various symmetric modes of propagation have nearly the same velocities for different n.

For the skew-symmetric modes of wave propagation (Fig. 2b), we observe the following: for the lowest mode (n = 0), the phase velocity profiles in the G – L and CT theories coincide for the wave number $\xi d \geq 2.2$ and as $\xi d \leq 2.2$ the phase velocity in the G – L theory is less than in the case of the CT theory; for n = 1, the phase velocity profiles

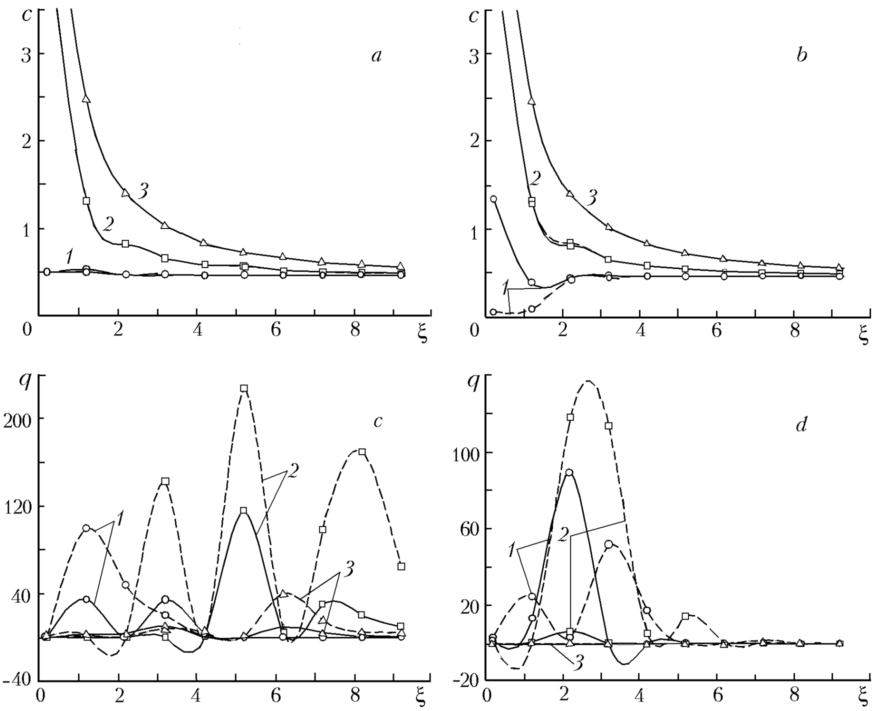

Fig. 2. Dependences of the phase velocity (a and b) and attenuation coefficient (c and d) on the wave number for the symmetric (a and c) and skew-symmetric (b and d) modes, various theories (solid and dashed curves correspond to the CT and G – L theories, respectively), and different values of n: 1) n = 0; 2) 1; 3) 2.

according to both theories coincide for any wave number except for those lying between 1.4 and 2.8, where the phase velocity in the G – L theory is slightly higher; for n = 2, the phase velocity according to both theories coincide.

The variation of the attenuation coefficient with the wave number for the symmetric and skew-symmetric modes is given in Fig. 2c and d, respectively. First we analyze the results of the G – L theory. For the lowest symmetric mode, the attenuation coefficient has a maximum of 99.36 at $\xi d = 1.2$ and then it approaches zero. For the first symmetric mode, the attenuation coefficient varies negligibly in the region $0.2 \le \xi d \le 2.2$; it has maxima equal to 142.3, 227.4, and 169.3 at $\xi d = 3.2, 5.2$, and 8.2, respectively, then it decreases to 64.73 at $\xi d = 9.2$. For the second symmetric mode, the magnitude of the attenuation coefficient varies slightly in the region $0.2 \le \xi d \le 5.2$, increases to 40.2 at $\xi d = 6.2$, and decreases to 5.13.

In the case of the CT theory, for the lowest symmetric mode the attenuation coefficient has maxima equal to 34.96 and 34.39 at $\xi d = 1.2$ and 3.2, respectively, and varies slightly in the region $4.2 \le \xi d \le 9.2$. For the first symmetric mode, the attenuation coefficient varies negligibly in the region $0.2 \le \xi d \le 4.2$; it has maxima of 115.7 and 30.4 at $\xi d = 5.2$ and 7.2, respectively, and then it decreases to 9.417 at $\xi d = 9.2$. For the second symmetric mode, the attenuation coefficient changes slightly.

For the lowest skew-symmetric mode, the magnitude of attenuation coefficient in the G — L theory has maxima equal to 24.57 and 51.72 at $\xi d = 1.2$ and 3.2, respectively, and it is characterized by a negligible variation in the region $4.2 \le \xi d \le 9.2$. For the first skew-symmetric mode there are two maxima, and the attenuation coefficient for the second skew-symmetric mode in the G – L theory is characterized by a negligible change.

For the lowest skew-symmetric mode, the attenuation coefficient according to the CT theory attains maxima of 89.57 at $\xi d = 2.2$, and changes slightly in the region $3.2 \le \xi d \le 9.2$. For the first skew-symmetric mode, this coefficient increases to 6.358 at $\xi d = 2.2$, and it is characterized by a small change in the region $3.2 \le \xi d \le 9.2$. The attenuation coefficient for the second skew-symmetric mode (n = 2) in the CT theory also changes weakly.

The effect of stress-free thermally insulated boundaries of the plate is quite pertinent and can be easily noticed from dispersion and attenuation curves in Fig. 2. The effect of micropolarity and microstretch can be seen from the comparison of the figures.

Conclusions. The propagation of free vibrations in a microstretch thermoelastic plate with two relaxation times that is subjected to stress-free thermally insulated and isothermal boundary conditions is investigated in the context of conventional coupled thermoelasticity (CT) and Green and Lindsay thermoelasticity (G — L) theories. The secular equations for the microstretch thermoelastic plate in a closed form and the boundary conditions for the symmetric and skew-symmetric modes in completely separate terms are derived. The secular equations for microstretch coupled thermoelastic, micropolar thermoelastic, and thermoelastic plates are shown to be particular cases of the equations derived. At short wavelength limits, the secular equations for the symmetric and skew-symmetric waves in stress free thermally insulated and isothermal microstretch thermoelastic plate reduce to the Rayleigh surface wave frequency equation. The symmetric and skew-symmetric wave modes are computed numerically and presented graphically. The theoretical and numerical results are found to be in close agreement.

NOTATION

c, nondimensional phase velocity; C^*, specificc heat; d, half-width of the plate; f_l, applied force density; $g-s$, microstretch stress; h, surface heat transfer coefficient; j, microinertia; j_0, microinertia of microelements; j_{lm}, microinertia tensor; K^*, thermal conductivity coefficient; m_{ij}, couple stress tensor; m_k, microstress tensor; q, attenuation coefficient of waves; t_{ij}, force stress tensor; T, temperature; T_0, uniform temperature; t, time; $\mathbf{u} = (u_1, u_2, u_3)$, displacement vector; v, propagation speed of waves; x, y, z, coordinates; α, β, γ, K, micropolar elasticity constants; α_{t1}, α_{t2}, coefficients of linear thermal expansion; $\alpha_0, \lambda_0, \lambda_1$, b_0, microstretch constants; δ_{ij}, Kronecker delta; ε_{lmn}, permutation symbol; λ, μ, elasticity constants; ω, nondimensional circular frequency; ω^*, characteristic frequency of the medium; $\phi = (\phi_1, \phi_2, \phi_3)$, microrotation vector; ϕ^*, scalar point microstretch function; ρ, density; τ_0, τ_1, thermal relaxation times; ξ, nondimensional wave number.

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